A Decomposition Theorem for Persistent Labelled Transition Systems

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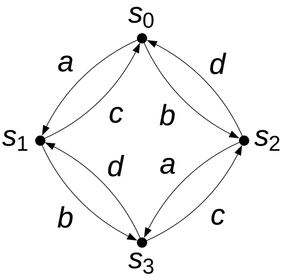
Labelled transition systems with initial state

$(S, \rightarrow, T, s_0)$ where
S are states
T are labels
$\rightarrow \subseteq (S \times T \times S)$ are the set of arcs
$s_0 \in S$ is an initial state.



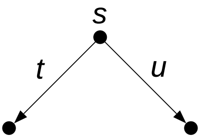
$$
\begin{array}{lcl}
S & = & \{s_0, s_1, s_2, s_3\} \\
T & = & \{a, b, c, d\} \\
\rightarrow & = & \{(s_0, a, s_1), (s_1, c, s_0), \\
 & & (s_0, b, s_2), (s_2, d, s_0), \\
 & & (s_1, b, s_3), (s_3, d, s_1), \\
 & & (s_2, a, s_3), (s_3, c, s_2)\}
\end{array}
$$

Reachability notation

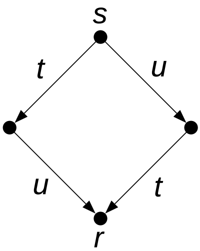
- $s[t\rangle$ if $\exists s' \in S : (s, t, s') \in \longrightarrow$
  t is enabled (activated, firable) in state s.
- $s[t\rangle s'$ if $(s, t, s') \in \longrightarrow$.
- Reachability:
  - $s[\varepsilon\rangle$ and $s[\varepsilon\rangle s$ are always true
  - $s[\sigma t\rangle$ iff there is some s'' with $s[\sigma\rangle s''$ and $s''[t\rangle$
    $s[\sigma t\rangle s'$ iff there is some s'' with $s[\sigma\rangle s''$ and $s''[t\rangle s'$.
- $[s\rangle$: set of states reachable from s.

Persistency of an lts $(S, \rightarrow, T, s_0)$

Persistency:



implies $\exists r$:



Our results are about the cyclic structure of a persistent lts.

Why are persistent systems interesting?

They cover a general notion of conflict-freeness.

Asynchronous Circuits Design People like them.

Every Petri net can be simulated by
a persistent net plus two non-persistent transitions.

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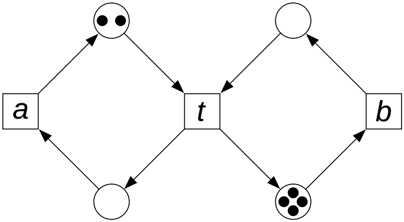
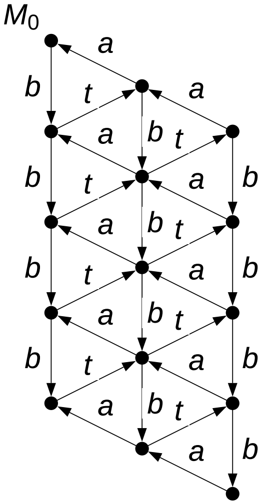
Marked graphs are persistent





Note:
All simple cycles have the same Parikh vectors

Parikh vectors

Let $\sigma = t_1 \dots t_n \in T^*$ be a finite sequence of labels.

Its Parikh vector is $\Psi(\sigma) : \begin{cases} T & \longrightarrow & \mathbb{N} \\ t & \longmapsto & \text{number of times } t \text{ occurs in } \sigma. \end{cases}$

Example: $\Psi(t_3 t_1 t_3)(t_1) = 1$
$\Psi(t_3 t_1 t_3)(t_2) = 0$
$\Psi(t_3 t_1 t_3)(t_3) = 2$

Permutations

- Two activated sequences $s[\sigma\rangle$ and $s[\sigma'\rangle$ arise from each other by a transposition if

$$\sigma = t_1 \dots t_k t\, t' \dots t_n \quad \text{and} \quad \sigma' = t_1 \dots t_k t' t \dots t_n .$$

- Two activated sequences $s[\sigma\rangle$ and $s[\sigma'\rangle$ are permutations of each other (from s, written $\sigma \equiv_s \sigma'$) if they arise out of each other through a sequence of transpositions.

Simple cycles

- A cycle from s is a firable sequence that reproduces s:

$$s[\sigma\rangle s.$$

- A cycle $s[\sigma\rangle s$ is simple

  if there is no permutation $\sigma \equiv_s \tau_1\tau_2$ with
  - τ_1 and τ_2 are nontrivial: $\tau_1 \neq \varepsilon \neq \tau_2$
  - $s[\tau_1\rangle s[\tau_2\rangle s$.

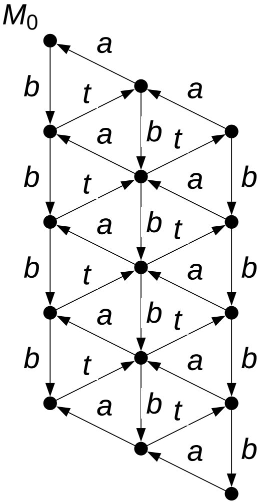
Example (non-simple vs. simple cycles)

- $M_0[btbata\rangle M_0$ is not a simple cycle

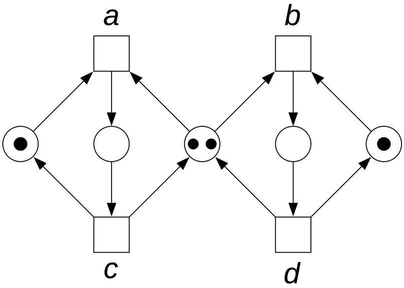
  because $btbata \equiv_{M_0} btabta$

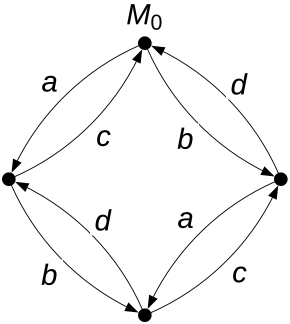
  and $M_0[bta\rangle M_0[bta\rangle M_0$.

- $M_0[bta\rangle M_0$ is a simple cycle.



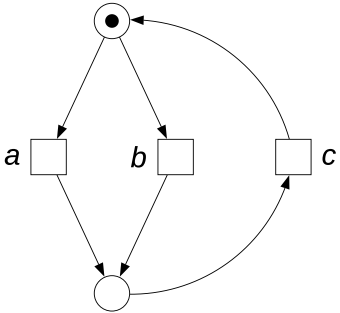
A persistent net which is not a marked graph

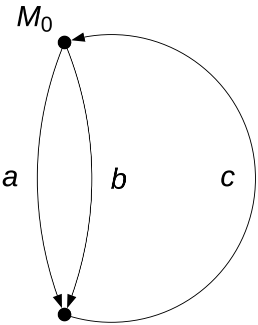




Note: All simple cycles have the same Parikh vectors or are transition-disjoint

A non-persistent net





Note: There are two simple cycles which do not have the same Parikh vectors and are not transition-disjoint

Some properties of an lts $(S, \longrightarrow, T, s_0)$

- finite if S and T are finite
- deterministic if $\forall s \in [s_0\rangle, t \in T : s[t\rangle s' \wedge s[t\rangle s'' \Rightarrow s' = s''$
- weakly periodic if for every $s_1 \in [s_0\rangle$, $\sigma \in T^*$, and

$$s_1[\sigma\rangle s_2[\sigma\rangle s_3[\sigma\rangle s_4[\sigma\rangle \ldots,$$

  either $\forall i, j \geq 1 \colon s_i = s_j$ or $\forall i, j \geq 1 \colon i \neq j \Rightarrow s_i \neq s_j$
- cycle-consistent if for all $\sigma \in T^*$,
  $\exists s \in [s_0\rangle \colon s[\sigma\rangle s$ implies $\forall s', s'' \in S \colon s'[\sigma\rangle s'' \Rightarrow s' = s''$

Reachability graphs of Petri nets are always deterministic, weakly periodic and cycle-consistent.

Main decomposition theorem

Let $(S, \rightarrow, T, s_0)$ be finite, deterministic,
weakly periodic, cycle-consistent, and persistent.

There exists a reachable state $\widetilde{s}$
and a finite set of label-disjoint simple cycles $\widetilde{s}[\rho_i\rangle\widetilde{s}$

such that:

for any reachable state s and for any cycle $s[\rho\rangle s$,
$\Psi(\rho) = \sum k_i \Psi(\rho_i)$ for some $k_i \geq 0$.

Note: This statement is wrong in the previous example.

Keller's theorem

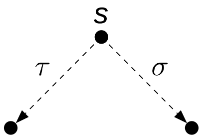
For label sequences τ and σ,

$$\tau \mathbin{\stackrel{\bullet}{-}} \varepsilon = \tau$$

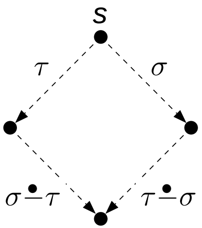
$$\tau \mathbin{\stackrel{\bullet}{-}} t = \begin{cases} \tau, \text{ if there is no label } t \text{ in } \tau \\ \text{the sequence obtained by erasing the leftmost } t \text{ in } \tau, \text{ otherwise} \end{cases}$$

$$\tau \mathbin{\stackrel{\bullet}{-}} (t\sigma) = (\tau \mathbin{\stackrel{\bullet}{-}} t) \mathbin{\stackrel{\bullet}{-}} \sigma.$$

Keller's theorem: if an lts is deterministic and persistent then



implies



Permutation lemma

Lemma: Let an lts be deterministic and persistent.
Let $s[\gamma\rangle$ and $s[\kappa\gamma\rangle$.
Then $s[\gamma\kappa'\rangle$ with $\Psi(\kappa) = \Psi(\kappa')$ and $\kappa\gamma \equiv_s \gamma\kappa'$.

Proof: By Keller's Theorem, $s[\gamma\rangle$ and $s[\kappa\gamma\rangle$ entail $s[\gamma(\kappa\gamma \stackrel{\bullet}{-} \gamma)\rangle$.
Put $\kappa' = \kappa\gamma \stackrel{\bullet}{-} \gamma$.
Then $\Psi(\kappa\gamma \stackrel{\bullet}{-} \gamma) = \Psi(\kappa)$, hence $\Psi(\kappa\gamma) = \Psi(\gamma\kappa')$.
$\kappa\gamma \equiv_s \gamma\kappa'$ follows since both sequences are activated at s.

Existence of home states

Proposition: Let an lts be finite, deterministic and persistent.
Then $\exists \widetilde{s} \in S \, \forall s \in [s_0\rangle : \widetilde{s} \in [s\rangle$.

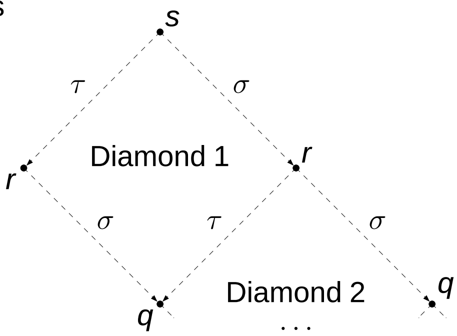
Proof: Let the set of reachable states be $\{s_0, \ldots, s_m\}$.
Put $\widetilde{s}_0 = s_0$.
Select for each i from 1 up to m some state $\widetilde{s}_i$ reachable from $\widetilde{s}_{i-1}$ and s_i, which exists by Keller's theorem.
Then put $\widetilde{s} = \widetilde{s}_m$.

Disjointness lemma

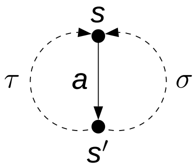
Lemma: Let an lts be finite, deterministic, weakly periodic, persistent. Let $s[\tau\rangle r$ and $s[\sigma\rangle r$ be two sequences with $s \neq r$. Then there is at least one label which occurs both in τ and in σ.

Proof: By contraposition, using Keller's Theorem.
If τ and σ are label-disjoint, then $\tau \stackrel{\bullet}{-} \sigma = \tau$ and $\sigma \stackrel{\bullet}{-} \tau = \sigma$.

The West and East corners of Diamond 1 and of Diamond 2 are the same by determinacy.
Thus: $s\,[\sigma\rangle\, r\,[\sigma\rangle\, q\,[\sigma\rangle\ \ldots$
By weak periodicity and $s \neq r$,
the set of reachable states is infinite.



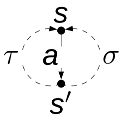
Lemma 1: about the uniqueness of simple cycles



Lemma 1: Let an lts be finite, deterministic, weakly periodic, cycle-consistent, and persistent.
Let $s[a\,\tau\rangle s$ and $s[a\,\sigma\rangle s$ with simple $s[a\,\tau\rangle s$ and $s[a\,\sigma\rangle s$.

Then $a\,\tau \equiv_s a\,\sigma$.

Proof outline of Lemma 1



- Using Keller's Theorem, $\tau \stackrel{\bullet}{-} \sigma = \varepsilon$ implies $\sigma \stackrel{\bullet}{-} \tau = \varepsilon$.
- By symmetry, there are two separate cases.
- **Case 1:** $\tau \stackrel{\bullet}{-} \sigma = \varepsilon = \sigma \stackrel{\bullet}{-} \tau$.
  Then $\Psi(\tau) = \Psi(\sigma)$, implying $\Psi(a\tau) = \Psi(a\sigma)$.
  Then also $a\,\tau \equiv_s a\,\sigma$.
- **Case 2:** $\tau \stackrel{\bullet}{-} \sigma \neq \varepsilon \neq \sigma \stackrel{\bullet}{-} \tau$. Then
  1. The sequences $\sigma \stackrel{\bullet}{-} \tau$ and $\tau \stackrel{\bullet}{-} \sigma$ are both activated at s, and when executed from s, they lead to the same state, say to $\widetilde{s}$.
  2. $\widetilde{s} \neq s$.

  By finiteness and the disjointness lemma, $\sigma \stackrel{\bullet}{-} \tau$ and $\tau \stackrel{\bullet}{-} \sigma$ have some label in common; contradiction.

Reversibility

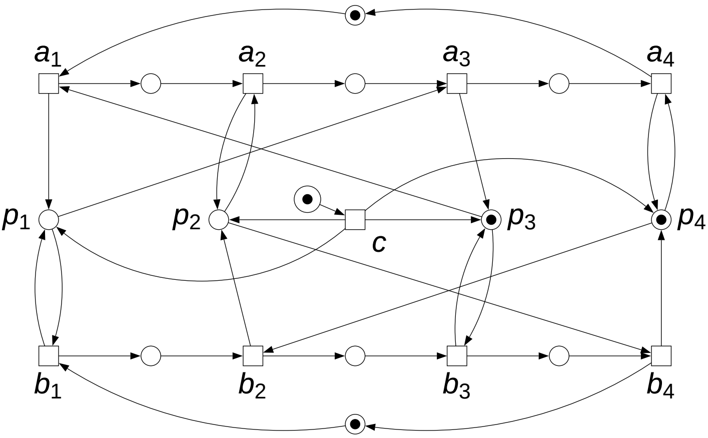
It would be nice to extend Lemma 1 in the following way:

> If two simple cycles $s_1[\tau\rangle s_1$ and $s_2[\sigma\rangle s_2$ have a label in common, then they are Parikh-equivalent.

However, this is true only for reversible lts.

An lts with initial state s_0 is called reversible if
$\forall s \in [s_0\rangle : s_0 \in [s\rangle$.

An non-reversible persistent net



Before firing c, $M_0[a_1b_1b_2a_2a_3b_3b_4a_4\rangle M_0$ is a simple cycle.

After firing $M_0[c\rangle M$,
$M[a_1a_2a_3a_4\rangle M$ and $M[b_1b_2b_3b_4\rangle M$ are simple cycles.

Hypersimple cycles

A cycle $s[\rho\rangle s$ is hypersimple if $\Psi(\rho)$ differs from $\Psi(\rho_1) + \Psi(\rho_2)$ for any two non-trivial cycles $s_1[\rho_1\rangle s_1$ and $s_2[\rho_2\rangle s_2$ from reachable markings s_1 and s_2.

In the previous example,

- $M[a1\, a2\, a3\, a4\rangle M$ and $M[b1\, b2\, b3\, b4\rangle M$ are hypersimple.
- $M_0[a1\, b1\, b2\, a2\, a3\, b3\, b4\, a4\rangle M_0$ is simple but not hypersimple.

At a home state, every simple cycle is hypersimple

Lemma: Let an lts be finite, deterministic, weakly periodic, cycle-consistent, and persistent. Let $\widetilde{s} \in [s_0\rangle$ be a home state. Then every simple cycle $\widetilde{s}[\rho\rangle\widetilde{s}$ is hypersimple.

Proof: Suppose $\Psi(\rho) = \Psi(\rho_1) + \Psi(\rho_2)$ for nontrivial cycles $s_1[\rho_1\rangle s_1$ and $s_2[\rho_2\rangle s_2$ from reachable states s_1 and s_2.
Because $\widetilde{s}$ is a home state, $s_1[\chi\rangle\widetilde{s}$ for some label sequence χ.
By the permutation lemma applied in s_1 with $\kappa = \rho_1$ and $\gamma = \chi$, $\Psi(\rho_1) = \Psi(\rho_1')$ for some cycle $\widetilde{s}[\rho_1'\rangle\widetilde{s}$.
By the definition of $\stackrel{\bullet}{-}$, $\rho_1' \stackrel{\bullet}{-} \rho = \varepsilon$ since $\Psi(\rho_1') \leq \Psi(\rho)$.
By Keller's Theorem, applied to $\widetilde{s}[\rho\rangle\widetilde{s}$ and $\widetilde{s}[\rho_1'\rangle\widetilde{s}$, $\widetilde{s}[\rho_1' \stackrel{\bullet}{-} \rho\rangle s$ and $\widetilde{s}[\rho \stackrel{\bullet}{-} \rho_1'\rangle s$ for some state s, with $\rho(\rho_1' \stackrel{\bullet}{-} \rho) \equiv_{\widetilde{s}} \rho_1'(\rho \stackrel{\bullet}{-} \rho_1')$.
As $\rho_1' \stackrel{\bullet}{-} \rho = \varepsilon$, $\widetilde{s} = s$ and $\Psi(\rho) = \Psi(\rho_1') + \Psi(\rho \stackrel{\bullet}{-} \rho_1')$.
Recalling that $\Psi(\rho) = \Psi(\rho_1) + \Psi(\rho_2)$, both $\Psi(\rho_1') = \Psi(\rho_1)$ and $\Psi(\rho \stackrel{\bullet}{-} \rho_1') = \Psi(\rho_2)$ differ from the null vector.
Now $\widetilde{s}[\rho_1'\rangle\widetilde{s}[\rho \stackrel{\bullet}{-} \rho_1'\rangle\widetilde{s}$, and therefore $\widetilde{s}[\rho\rangle\widetilde{s}$ is not a simple cycle.

Lemma 2: adapting Lemma 1

Lemma 2: Let an lts be finite, deterministic, weakly periodic, cycle-consistent, and persistent.

Let s, s' be reachable states
and $s[\tau\rangle s$ and $s'[\sigma\rangle s'$ be two hypersimple cycles.

If some label a occurs in both cycles, then $\Psi(\tau) = \Psi(\sigma)$.

Proof of Lemma 2 (part 1 of 2)

By Keller's Theorem, there exist a state s'' and two label sequences ξ and χ such that $s[\xi\rangle s''$ and $s'[\chi\rangle s''$.
By the permutation lemma applied in s with $\gamma = \xi$ and $\kappa = \tau$, there exists a label sequence τ' such that $s''[\tau'\rangle s''$ and $\Psi(\tau) = \Psi(\tau')$.
By the permutation lemma applied in s' with $\gamma = \chi$ and $\kappa = \sigma$, there exists a label sequence σ' such that $s''[\sigma'\rangle s''$ and $\Psi(\sigma) = \Psi(\sigma')$.
Let $\tau' = \tau_1' t \tau_2'$ and $\sigma' = \sigma_1' t \sigma_2'$ such that t occurs neither in τ_1' nor in σ_1', and let r and r' be the two states such that $s''[\tau_1'\rangle r$ and $s''[\sigma_1'\rangle r'$, respectively.
By Keller's Theorem, applied to $s''[\tau_1'\rangle r$ and $s''[\sigma_1'\rangle r'$, there exists a state r'' such that $r[\sigma_1' \stackrel{\bullet}{-} \tau_1'\rangle r''$ and $r'[\tau_1' \stackrel{\bullet}{-} \sigma_1'\rangle r''$.

Proof of Lemma 2 (part 2 of 2)

By the permutation lemma applied in r with $\gamma = \sigma'_1 \stackrel{\bullet}{-} \tau'_1$ and $\kappa = t\tau'_2\tau'_1$, there exists a label sequence τ'' such that $r''[\tau''\rangle r''$ and $\Psi(\tau'') = \Psi(t\tau'_2\tau'_1) = \Psi(\tau)$.
Similarly, there exists a label sequence σ'' such that $r''[\sigma''\rangle r''$ and $\Psi(\sigma'') = \Psi(t\sigma'_2\sigma'_1) = \Psi(\sigma)$.
Now $r[t\rangle$, $r[\sigma'_1 \stackrel{\bullet}{-} \tau'_1\rangle r''$, and label t does not occur in $\sigma'_1 \stackrel{\bullet}{-} \tau'_1$ since it does not occur in σ'_1.
By persistency, $r''[t\rangle\widetilde{r}$ for some state $\widetilde{r}$.
As $\Psi(t) \leq \Psi(\tau'') = \Psi(\tau)$, $t \stackrel{\bullet}{-} \tau'' = \varepsilon$.
By Keller's Theorem, applied to $r''[\tau''\rangle r''$ and $r''[t\rangle\widetilde{r}$, $\widetilde{r}[\tau'' \stackrel{\bullet}{-} t\rangle r''$.
As the Parikh vector of $r''[t\rangle\widetilde{r}[\tau'' \stackrel{\bullet}{-} t\rangle r''$ is equal to $\Psi(\tau'') = \Psi(\tau)$, this cycle is hypersimple.
Similarly, one can construct a hypersimple cycle $r''[t\rangle\widetilde{r}[\sigma'' \stackrel{\bullet}{-} t\rangle r''$.
As every hypersimple cycle is simple and both cycles start with t from r'', Lemma 1 applies, entailing $t(\tau'' \stackrel{\bullet}{-} t) \equiv_{s''} t(\sigma'' \stackrel{\bullet}{-} t)$ and hence $\Psi(\tau) = \Psi(\tau'') = \Psi(\sigma'') = \Psi(\sigma)$.

Putting the pieces together

Let an lts be finite, deterministic, weakly periodic, cycle-consistent, and persistent.

There exists a reachable state $\widetilde{s}$
and a finite set of label-disjoint simple cycles $\widetilde{s}[\rho_i\rangle\widetilde{s}$

such that:

for any reachable state s and for any cycle $s[\rho\rangle s$,
$\Psi(\rho) = \sum k_i \Psi(\rho_i)$ for some $k_i \geq 0$.

Roadmap of the proof: Choose some home state $\widetilde{s}$.
Push $s[\rho\rangle s$ to a Parikh-equivalent cycle $\widetilde{s}[\rho'\rangle\widetilde{s}$.
Permute and decompose $\widetilde{s}[\rho'\rangle\widetilde{s}$ into a sequence of simple cycles through $\widetilde{s}$.
Any simple cycle $\widetilde{s}[\rho\rangle\widetilde{s}$ is hypersimple.
By Lemma 2, two simple cycles through $\widetilde{s}$ are either transition-disjoint or Parikh equivalent.

The special case of reversible Petri nets

For reversible, bounded and persistent nets

- the notions of simplicity and hypersimplicity coincide
- and every reachable marking is a home marking.

Decomposition corollary:

Let N be reversible, bounded, and persistent.

There is a finite set $\mathcal{B}$ of semipositive T-invariants such that any two of them are transition-disjoint and every cycle $M[\rho\rangle M$ in the reachability graph decomposes up to permutations to some sequence of cycles $M[\rho_1\rangle M[\rho_2\rangle M \ldots [\rho_n\rangle M$ with all Parikh vectors $\Psi(\rho_i)$ in $\mathcal{B}$.

Difference to the decomposition theorem:
$M[\rho\rangle M$ can be decomposed already at M.

A consequence of the decomposition corollary

Every bounded, persistent and reversible Petri net N
whose unique minimal integral basis $\mathcal{B}$ satisfies $|\mathcal{B}| = n$
can be viewed (up to reachability graph isomorphism)
as the $\oplus$ of n bounded, persistent and reversible Petri nets N_i
whose unique minimal integral bases $\mathcal{B}_i$ satisfy $|\mathcal{B}_i| = 1$.

(Not an immediate corollary.)

The special case of marked graphs

- If there is some nontrivial cycle in the reachability graph of a weakly connected marked graph, then it is automatically reversible.
  Hence we have a unique basis $\mathcal{B}$ as in the decomposition corollary.
- The vector assigning the number 1 to every transition is the only member of $\mathcal{B}$.
  Thus, all Parikh vectors of cycles are multiples of $(1, \ldots, 1)$ (recovering a well-known result).

The interest in this research may lie...

- ...in the proofs of Lemmas 1 and 2, both of which are non-trivial applications of Keller's fundamental theorem...
- ...in that it describes a rather nice property of the class of transition systems in question, which may have several other consequences that still need to be looked at...
- ...such as, perhaps, separability.

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Separability

Origin and application: Workflow verification [van Hee et al.]

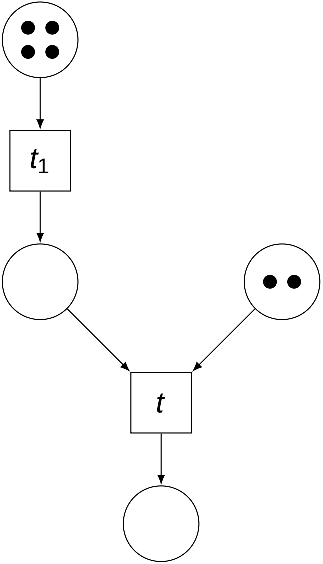
Let $k \in \{1, 2, 3, \ldots\}$ be a number.
Let M_0 be an initial marking of a net N such that every place has a multiple of k tokens (0 or k or $2k$ or $\ldots$).
(N, M_0) is called *k-separable* if, for every firable sequence $M_0[\sigma\rangle$, there are $\sigma_1, \ldots, \sigma_k$ such that

$$\forall j, 1{\leq}j{\leq}k: \ (\frac{1}{k}{\cdot}M_0)[\sigma_j\rangle \quad \text{and} \quad Parikh(\sigma) = \sum_{j=1}^{k} Parikh(\sigma_j).$$

The vector $Parikh(\sigma)$, for a sequence σ of transitions, counts the number of each transition in σ.

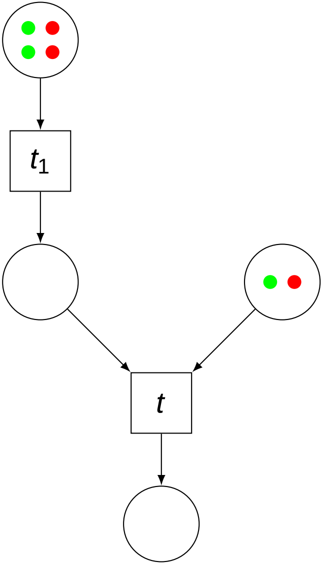
Theorem: Marked graphs are separable.
They can thus be viewed as independent copies (direct sums) of k safe marked graphs $\rightsquigarrow$ reduced state space.

$k = 2$, separable



$t_1\ t\ t_1\ t$

$k = 2$, a separation

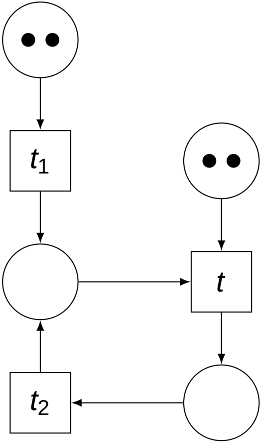


$t_1 \; t \; t_1 \; t$

$t_1 \; t \; t_1 \; t$ ↯

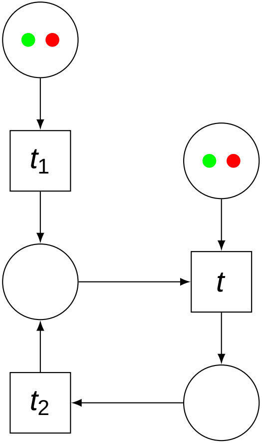
$t_1 \; t \; t_1 \; t$ ✓

$k = 2$, not separable



$t_1\, t\, t_2\, t$

$k = 2$, no separation possible



$t_1\ t\ t_2\ t$

$t_1\ t\ t_2\ t\ \lightning$

$t_1\ t\ t_2\ t\ \lightning$

Open question

Are

bounded, reversible and persistent Petri nets

separable or not?

As a consequence of the
consequence of the decomposition corollary,
we need only consider the case that there is
a single minimal realisable T-invariant.